Published by Institute of Physics Publishing for SISSA

Received: *April 4, 2006*
Revised: *June 16, 2006*
Accepted: *July 13, 2006*
Published: *August 16, 2006*

The SU(1|2) and SU(2|2) sectors from superstrings in $AdS_5 \times S^5$

Shijong Ryang

Department of Physics, Kyoto Prefectural University of Medicine
Taishogun, Kyoto 603-8334 Japan
E-mail: ryang@koto.kpu-m.ac.jp

Abstract: For the κ-symmetry gauge fixed superstring action in $AdS_5 \times S^5$ we consider the fermionic fluctuations over a circular bosonic string background with two angular momenta and two winding numbers in S^5. The SU(2)-type redefinitions of fermionic fields and the first-string limit generate a truncated fermionic action for the SU(1|2) sector. It is expressed in a two-dimensional Lorentz-invariant form of a massive Dirac fermion and the plane-wave spectrum for the fermionic excitations is derived. The fermionic spectrum for the SU(2|2) sector is also analyzed.

Keywords: Penrose limit and pp-wave background, AdS-CFT Correspondence.

Contents

1. Introduction

The AdS/CFT correspondence [1] has been explored beyond the supergravity approximation. Inspired from the solvability of the string theory in the pp-wave background of $AdS_5 \times S^5$ [2], it has been proposed that the energies of specific free massive string excited states can be matched with the perturbative scaling dimensions of gauge invariant near-BPS operators with large R-charge in the BMN limit for the $\mathcal{N} = 4$ SU(N) super Yang-Mills (SYM) theory [3]. The BMN result has been reproduced by the semiclassical quantization of nearly point-like string with large angular momentum along a central circle in S^5 [4]

The energies of various semiclassical extended string configurations with several large angular momenta in $AdS_5 \times S^5$ have been shown in [5 − 8] to match with the anomalous dimensions of the corresponding long SYM non-BPS operators, which can be computed by using the Bethe ansatz [9] for diagonalization of the dilatation operator [10 − 12], that is represented by a Hamiltonian of an integrable spin chain.

From the view point of integrability the gauge/string duality has been further confirmed by verifying the equivalence between the classical string Bethe equation for the classical $AdS_5 \times S^5$ string sigma model and the Bethe equation for the spin chain [13, 14]. Combining the classical string Bethe ansatz and the all-loop gauge theory asymptotic Bethe ansatz [15], it has been shown that a novel Bethe ansaz, namely, the quantum string Bethe ansatz for the SU(2) sector has been constructed [16] such that it generates the classical spinning strings, the $\lambda^{1/4}$ strong coupling asymptotics and the $1/J$ energy corrections of arbitrary M-impurity BMN states whose special $M = 2, 3$ cases agree with the results of direct light-cone gauge quantization of the interacting string theory in the near plane-wave background [17]. The gauge theory asymptotic Bethe ansatz for the SU(2) sector has been shown to arise as an approximation to the Hubbard model [18]. The quantum string Bethe ansatz has been generalized by constructing the S matrices to the other sectors such as

SL(2), SU(1|1) [19] and the full PSU(2,2|4) [20]. For the SU(1|1) sector the dilatation operator at one-loop has been shown to coincide with the Hamiltonian of the free lattice fermion [21]. The S matrices leading to the asymptotic Bethe equations have been investigated for the SU(1|2) and SU(2|2) sectors [22] and the two-loop dilatation operator for the SU(1,1|2) sector has been constructed [23].

On the other hand there have been various studies of comparing the quantum worldsheet corrections to spinning string solutions in $AdS_5 \times S^5$ with the finite size corrections to the Bethe equations [24, 25].

The gauge/string duality has been also presented at the level of equations of motion [26] and at the level of effective action [27] where an interpolating spin chain sigma model action constructed by taking the continuum limit of the spin chain in the coherent basis for the SU(2) sector is also reproduced by taking some fast-string limit of the string action. The latter approach has been extended to the whole SO(6) and its compact subgroups [28–30] and non-compact SL(2) [30, 31]. Based on the spin chain sigma model for the SU(2) sector the $1/J$ and $1/J^2$ energy corrections to the plane-wave state and the circular and folded string states have been computed [32]. The supersymmetric extensions have been performed for SU(1|3) [33], SU(1,1|1) [34], SU(1,1|2) [35] and SU(2|3) [33, 36]. In [33–35] the first-string limit has been taken for certain subsectors of the covariant κ-symmetric superstring action in $AdS_5 \times S^5$ [37] constructed as a 2d sigma model on the coset superspace PSU(2,2|4)/[SO(1,4) $\times$ SO(5)], while in [36] it has been taken for a subsector of the light-cone κ-symmetry gauge fixed superstring action [38] expressed in terms of the light-cone supercoset coordinates in the SU(3) $\times$ U(1) invariant form.

From the superstring sigma model action in $AdS_5 \times S^5$ expressed in terms of the Z_4-graded current of the PSU(2,2|4)/[SO(1,4) $\times$ SO(5)] supercoset [39], the truncations to the SU(1|1) sector have been performed by choosing a phase-space uniform gauge $t = \tau, p_\phi = J$ [40] where p_ϕ is the canonical momentum conjugate to the angle variable ϕ for a central circle in S^5 and a uniform light-cone gauge [41], where the BMN spectrum for fermions is presented and the $1/J$ correction to the M-impurity plane-wave state agrees with the result of [42]. For the former gauge choice the two complex fermions are arranged into a single world-sheet Dirac fermion so that the reduced action shows a non-trivial 2d Lorentz-invariant interacting theory of massive Dirac fermion, while for the latter gauge choice the reduced theory becomes free and the femionic fluctuation spectra over both a point-like string with no winding numbers and an extended string wound around the ϕ direction are computed. For the former truncation to the SU(1|1) sector the exact S-matrix has been computed to give the Bethe ansatz solution [43].

In ref. [36] the SU(1|1) sector has been extracted from the SU(3) $\times$ U(1) invariant superstring action, where the fermionic action for the quadratic fermionic fluctuation over the point-like bosonic string background with a large angular momentum becomes a non-relativistic expansion form of an action for a massive 2d relativistic fermion. In order to see the effect of the bosonic background on the fermionic fluctuation we will consider the SU(1|2) sector. We will study the fermionic fluctuation around the non-point-like circular string background specified by two angular momenta and two winding numbers in S^5, and take the large limit of the total angular momentum. The fermionic action simplified by taking the first-string limit will be expressed in an explicitly 2d relativistic manner by a massive world-sheet Dirac fermion and shown to have the plane-wave spectrum. The fermionic spectrum for the SU(2|2) sector will be discussed.

2. SU(3) $\times$ U(1) invariant superstring action

We consider the superstring in $AdS_5 \times S^5$ space-time with metric $ds^2 = e^{2\phi}dx^a dx^a + d\phi d\phi + dX^M dX^M$, $X^M X^M = 1$ $(a = 0, \ldots, 3; M = 1, \ldots, 6)$. In terms of the light-cone coordinates on the PSU(2,2|4)/[SO(1,4) $\times$ SO(5)] supercoset the full Lagrangian $\mathcal{L} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{WZ}$ in the fermionic light-cone κ-symmetry gauge is constructed as [38]

$$\begin{aligned}
\mathcal{L}_{\text{kin}} = & -\frac{1}{2}\sqrt{g}g^{\mu\nu}[2e^{2\phi}(\partial_\mu x^+ \partial_\nu x^- + \partial_\mu x \partial_\nu \bar{x}) + \partial_\mu \phi \partial_\nu \phi + \partial_\mu X^M \partial_\nu X^M] \\
& -\frac{i}{2}\sqrt{g}g^{\mu\nu}e^{2\phi}\partial_\mu x^+[\theta^A \partial_\nu \theta_A + \theta_A \partial_\nu \theta^A + \eta^A \partial_\nu \eta_A + \eta_A \partial_\nu \eta^A] \\
& -i\sqrt{g}g^{\mu\nu}e^{2\phi}\partial_\mu x^+ X^N \partial_\nu X^M \eta_A \rho^{MNA}{}_B \eta^B \\
& +\frac{1}{2}\sqrt{g}g^{\mu\nu}e^{4\phi}\partial_\mu x^+ \partial_\nu x^+[(\eta^A \eta_A)^2 + (X^N \eta_A \rho^{MNA}{}_B \eta^B)^2], \\
\mathcal{L}_{WZ} = & \, \epsilon^{\mu\nu}e^{2\phi}\partial_\mu x^+ X^M(\eta^A \rho^M_{AB} \partial_\nu \theta^B + \eta_A \rho^{MAB} \partial_\nu \theta_B) \\
& +i\sqrt{2}\epsilon^{\mu\nu}e^{3\phi}\partial_\mu x^+ X^M(\partial_\nu \bar{x} \eta_A \rho^{MAB} \eta_B - \partial_\nu x \eta^A \rho^M_{AB} \eta^B),
\end{aligned} \tag{2.1}$$

where $g_{\mu\nu}$ $(\mu = 0, 1)$ is a world-sheet metric with signature $(-, +)$ and $g = -\det g_{\mu\nu}$. The Poincare coordinates of AdS_5 are chosen by

$$x^{\pm} = \frac{1}{\sqrt{2}}(x^3 \pm x^0), \qquad x = \frac{1}{\sqrt{2}}(x^1 + ix^2), \qquad \bar{x} = \frac{1}{\sqrt{2}}(x^1 - ix^2) \tag{2.2}$$

and the radial direction ϕ, while S^5 is parametrized by a unit 6-vector X^M so that the constraint $X^M X^M = 1$ should be imposed with a Lagrange multiplier Λ. This Lagrangian has manifest SU(4) symmetry where the $4 + 4$ complex fermionic fields θ_A, η_A with $A = 1, 2, 3, 4$ transform in the fundamental representation of SU(4) and $\theta^A = \theta^\dagger_A, \eta^A = \eta^\dagger_A$. The 4×4 matrices ρ^M are "off-diagonal" blocks of the SO(6) gamma-matrices in the chiral representation and $\rho^{MN} = -\rho^{[M}\rho^{*N]}$. There exist only quadratic and quartic fermionic terms which are associated with special symmetries of the $AdS_5 \times S^5$ background. The $(\eta^A \eta_A)^2$ term reflects the curvature of the background, and the $(X^N \eta_A \rho^{MNA}{}_B \eta^B)^2$ term is interpreted as the coupling to the R-R 5-form background. The fermionic fields θ_A are related to the linearly realized supersymmetry of the super conformal algebra PSU(2,2|4), and the η_A fields are associated with the non-linearly realized superconformal symmetry.

In ref. [36] the bosonic AdS_5 Poincare coordinates are replaced by the global AdS_5 ones and in order to choose the conformal gauge for the 2d metric the following ansatz corresponding to the global AdS_5 time $t = \kappa\tau + \cdots$ where dots indicate possible fermionic terms, is taken for the bosonic AdS_5 fields

$$e^\phi = \cos\kappa\tau, \quad x^+ = \frac{\tan\kappa\tau}{\sqrt{2}}, \quad x^- = -\frac{\tan\kappa\tau}{\sqrt{2}} + f(\tau, \sigma), \quad x = \bar{x} = 0. \tag{2.3}$$

The ϕ equation of motion restricts on the allowed fermionic configuration to determine $\partial_0 f$, while one of the two conformal constraints determines $\partial_1 f$.

In the SU(2|3) sector on the SYM side the gauge invariant operators consist of the 3 chiral complex combinations of 6 scalars on which the SO(6) R-symmetry acts and the two spinor components of the gluino Weyl fermion which are singlets under the Cartan $[\mathrm{U}(1)]^3$ subgroup of SO(6). In order to extract the SU(2|3) sector the 3 chiral bosonic fields X_i are introduced by $X_i \equiv X_{2i-1} + iX_{2i},\ i = 1, 2, 3$ and the SU(4) fermions are splitted in $3 + 1$ way as $\eta_A \equiv (\eta_i, \eta),\ \theta_A \equiv (\theta_i, \theta), i = 1, 2, 3$. The two SU(3) singlet fields $\eta \equiv \eta_4, \theta \equiv \theta_4$ are related with the two fermions in the SU(2|3) sector.

The fermionic part of the Lagrangian (2.1) can be expressed in the manifestly $\mathrm{SU}(3) \times \mathrm{U}(1)$ invariant form through the ansatz (2.3) as $\mathcal{L}_F = \mathcal{L}_{2F} + \mathcal{L}_{4F}$ where the quadratic terms are

$$\begin{aligned}
\mathcal{L}_{2F} = \frac{\kappa}{\sqrt{2}}[& i\eta^i\partial_0\eta_i + i\bar{\eta}\partial_0\eta + i\theta^i\partial_0\theta_i + i\bar{\theta}\partial_0\theta + \epsilon_{ijk}\eta^i\partial_1\theta^j X^k - \epsilon^{ijk}\eta_i\partial_1\theta_j X_k \\
& +\eta^i\partial_1\bar{\theta}X_i - \eta_i\partial_1\theta X^i + \partial_1\theta^i\bar{\eta}X_i - \partial_1\theta_i\eta X^i \\
& -i(X^i\partial_0 X_j - X_j\partial_0 X^i)\eta_i\eta^j - iX^i\partial_0 X_i(\eta^j\eta_j - \bar{\eta}\eta) \\
& -i(\epsilon^{ijk}X_j\partial_0 X_k\eta_i\bar{\eta} - \epsilon_{ijk}X^j\partial_0 X^k\eta\eta^i)]
\end{aligned} \tag{2.4}$$

and the quartic terms are

$$\begin{aligned}
\mathcal{L}_{4F} = -\left(\frac{\kappa}{\sqrt{2}}\right)^2 & [3\eta^i\eta_i\bar{\eta}\eta - 4X_i\eta^i X^j\eta_j\bar{\eta}\eta + 4\eta_i X^i\eta^j X_j\eta_k\eta^k \\
& +2\epsilon_{ijk}\eta^i\eta^j X^k\eta_l X^l\eta + 2\epsilon^{ijk}\eta_i\eta_j X_k\eta^l X_l\bar{\eta}],
\end{aligned} \tag{2.5}$$

where $X^i = X_i^*, \eta^i = \eta_i^\dagger, \theta^i = \theta_i^\dagger, \bar{\eta} = \eta^\dagger, \bar{\theta} = \theta^\dagger$ and there are no gamma-matrices. The coupling terms including the σ-derivative in (2.4) originate in the Wess-Zumino part $\mathcal{L}_{WZ}$ of (2.1), while the coupling terms including the τ-derivative in the form $X\partial_0 X\eta\eta$ are due to the quadratic terms proportional to $\eta_A\rho^{MNA}{}_B\eta^B$ in the kinetic part $\mathcal{L}_{\mathrm{kin}}$ of (2.1). The bosonic part of the Lagrangian (2.1) is also expressed as

$$\mathcal{L}_B = -\frac{1}{2}\partial^\mu X_i^*\partial_\mu X_i + \frac{1}{2}\Lambda(X_i^* X_i - 1). \tag{2.6}$$

3. Fermionic fluctuations over a circular string with two equal spins

Since the starting fermionic Lagrangian $\mathcal{L}_F$ is expressed in the manifestly $\mathrm{SU}(3) \times \mathrm{U}(1)$ form, the truncations to the SU(2|3) and SU(1|1) sectors are facilitated [36]. There are two possible consistent truncations A and B with the bosonic fields from $AdS_3 \times S^3$ and fermions suitably chosen as

$$\mathrm{A}:\ (X_1, X_2; \theta, \theta_3, \eta_1, \eta_2) \neq 0, \qquad (x, X_3; \eta, \eta_3, \theta_1, \theta_2) = 0, \tag{3.1}$$

or, alternatively,

$$\mathrm{B}:\ (X_1, X_2; \eta, \eta_3, \theta_1, \theta_2) \neq 0, \qquad (x, X_3; \theta, \theta_3, \eta_1, \eta_2) = 0. \tag{3.2}$$

This "SU(1|2)" string theory sector is considered to be related with the SU(1|2) gauge theory sector. Further restricting to S^1 inside S^5 we have the following two truncations that are associated with the SU(1|1) gauge theory sector

$$\mathrm{A}' : \qquad (X_1; \theta, \eta_1) \neq 0, \tag{3.3}$$

$$\mathrm{B}' : \qquad (X_1; \eta, \theta_1) \neq 0, \tag{3.4}$$

where the other bosonic and fermionic fields are switched off respectively. Some solitonic classical solutions for these subsectors were constructed to include the fermionic semi-classical contribution as the generalization of the bosonic spinning string solutions [36]. For the "SU(1|1)" string theory sectors A' and B' the non-relativistic actions of BMN-type massive fermionic fluctuations were constructed from the $\mathrm{SU}(3) \times \mathrm{U}(1)$ invariant Lagrangian (2.4), (2.5) and (2.6) in the point-like bosonic string background by integrating extra fermions η_1 and θ_1 respectively.

As an extended string background we prepare a circular string solution specified by the winding number m with large two equal spins $\mathcal{T}_1 = \mathcal{T}_2 = \mathcal{T}/2$, which is expressed as

$$X_1 = \frac{1}{\sqrt{2}} e^{i\omega\tau - im\sigma}, \qquad X_2 = \frac{1}{\sqrt{2}} e^{i\omega\tau + im\sigma} \tag{3.5}$$

with $\omega = \mathcal{T}$ [7]. Its energy is characterized by $\mathcal{E}^2 = \kappa^2 = \mathcal{T}^2 + m^2$. This string background is determined from the leading order relations of the conformal constraints where there are no fermionic semiclassical contributions. We consider the fermionic excitation representing a small perturbation over the circular bosonic string background. This configuration is mapped to the long SYM operator which is composed of the large and same number of two comlex scalar fields and only a few fermions. It is convenient to rescale the fermionic fields as

$$\begin{pmatrix} \eta_i \\ \eta^i \end{pmatrix} \to \alpha \begin{pmatrix} \eta_i \\ \eta^i \end{pmatrix}, \; \begin{pmatrix} \eta \\ \bar{\eta} \end{pmatrix} \to \alpha \begin{pmatrix} \eta \\ \bar{\eta} \end{pmatrix}, \; \begin{pmatrix} \theta_i \\ \theta^i \end{pmatrix} \to \alpha \begin{pmatrix} \theta_i \\ \theta^i \end{pmatrix}, \; \begin{pmatrix} \theta \\ \bar{\theta} \end{pmatrix} \to \alpha \begin{pmatrix} \theta \\ \bar{\theta} \end{pmatrix} \tag{3.6}$$

with $\alpha = (\sqrt{2}/\kappa)^{1/2}$ in order to absorb the overall factors $\kappa/\sqrt{2}$ in $\mathcal{L}_{2F}$ and $(\kappa/\sqrt{2})^2$ in $\mathcal{L}_{4F}$.

Here we start to consider the case A. Plugging the bosonic background (3.5) into the quadratic Lagrangian (2.4) with $\kappa/\sqrt{2} = 1$ we have

$$\begin{aligned} \mathcal{L}_{2F} = {} & i \sum_{i=1}^{2} \eta^i \partial_0 \eta_i + i\theta^3 \partial_0 \theta_3 + i\bar{\theta}\partial_0\theta \\ & - \frac{e^{-i\omega\tau}}{\sqrt{2}} (e^{-im\sigma}\eta^1 - e^{im\sigma}\eta^2)\partial_1\theta^3 + \frac{e^{i\omega\tau}}{\sqrt{2}} (e^{im\sigma}\eta_1 - e^{-im\sigma}\eta_2)\partial_1\theta_3 \\ & + \frac{e^{i\omega\tau}}{\sqrt{2}} (e^{-im\sigma}\eta^1 + e^{im\sigma}\eta^2)\partial_1\bar{\theta} - \frac{e^{-i\omega\tau}}{\sqrt{2}} (e^{im\sigma}\eta_1 + e^{-im\sigma}\eta_2)\partial_1\theta \\ & + \omega(e^{2im\sigma}\eta_1\eta^2 + e^{-2im\sigma}\eta_2\eta^1). \end{aligned} \tag{3.7}$$

The quartic Lagrangian (2.5) becomes

$$\mathcal{L}_{4F} = -4\eta_1\eta^1\eta_2\eta^2. \tag{3.8}$$

In view of the terms including the σ-derivative in (3.7) which are associated with the Wess-Zumino part we introduce a change of fermionic variables

$$\begin{pmatrix} \eta_- \\ \eta_+ \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{im\sigma} & -e^{-im\sigma} \\ e^{im\sigma} & e^{-im\sigma} \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}, \tag{3.9}$$

which is an SU(2)-type rotation. We use $\omega = \mathcal{T}$ and the inverse relation of (3.9) to rewrite (3.7) as

$$\begin{aligned} \mathcal{L}_{2F} &= i\eta^+\partial_0\eta_+ + i\bar{\theta}\partial_0\theta + e^{i\mathcal{T}\tau}\eta^+\partial_1\bar{\theta} - e^{-i\mathcal{T}\tau}\eta_+\partial_1\theta - \mathcal{T}\eta^+\eta_+ \\ &\quad + i\eta^-\partial_0\eta_- + i\theta^3\partial_0\theta_3 - e^{-i\mathcal{T}\tau}\eta^-\partial_1\theta^3 + e^{i\mathcal{T}\tau}\eta_-\partial_1\theta_3 + \mathcal{T}\eta^-\eta_-, \end{aligned} \tag{3.10}$$

where $\eta^+ = \eta_+^\dagger, \eta^- = \eta_-^\dagger$ and the σ-dependent exponential phase factors $e^{\pm 2im\sigma}$ in the mixed terms in (3.7) have been eliminated. Under the SU(2)-type rotation (3.9) the kinetic terms remain the canonical forms. In the expression $\mathcal{L}_{2F}$ (3.7) the fermionic field η_1 is coupled with η^2, and η_2 with η^1, while the transformed one (3.10) is simplified such that η_+ is separated from η_-. The quartic expression (3.8) is changed into

$$\mathcal{L}_{4F} = -4\eta^+\eta_+\eta^-\eta_-. \tag{3.11}$$

Making the scalings of θ and θ_3 as $\theta \to e^{i\mathcal{T}\tau}\theta, \theta_3 \to e^{-i\mathcal{T}\tau}\theta_3$ to remove the time-dependent exponential phase factors in the mixed terms, we have a symmetric expression

$$\begin{aligned} \mathcal{L}_{2F} &= i\eta^+\partial_0\eta_+ + i\bar{\theta}\partial_0\theta + \eta^+\partial_1\bar{\theta} - \eta_+\partial_1\theta - \mathcal{T}(\bar{\theta}\theta + \eta^+\eta_+) \\ &\quad + i\eta^-\partial_0\eta_- + i\theta^3\partial_0\theta_3 - (\eta^-\partial_1\theta^3 - \eta_-\partial_1\theta_3) + \mathcal{T}(\theta^3\theta_3 + \eta^-\eta_-). \end{aligned} \tag{3.12}$$

Let us turn to the case B. The quadratic Lagrangian (2.4) in the bosonic background (3.5) is written by

$$\begin{aligned} \mathcal{L}_{2F} &= i\sum_{i=1}^{2}\theta^i\partial_0\theta_i + i\eta^3\partial_0\eta_3 + i\bar{\eta}\partial_0\eta \\ &\quad + \frac{e^{-i\omega\tau}}{\sqrt{2}}\eta^3(e^{-im\sigma}\partial_1\theta^1 - e^{im\sigma}\partial_1\theta^2) - \frac{e^{i\omega\tau}}{\sqrt{2}}\eta_3(e^{im\sigma}\partial_1\theta_1 - e^{-im\sigma}\partial_1\theta_2) \\ &\quad + \frac{e^{i\omega\tau}}{\sqrt{2}}(e^{-im\sigma}\partial_1\theta^1 + e^{im\sigma}\partial_1\theta^2)\bar{\eta} - \frac{e^{-i\omega\tau}}{\sqrt{2}}(e^{im\sigma}\partial_1\theta_1 + e^{-im\sigma}\partial_1\theta_2)\eta \\ &\quad + \omega(\eta^3\eta_3 - \bar{\eta}\eta), \end{aligned} \tag{3.13}$$

while the quartic one (2.5) is

$$\mathcal{L}_{4F} = -3\eta^3\eta_3\bar{\eta}\eta. \tag{3.14}$$

The expression (3.13) also suggests the following SU(2)-type rotation

$$\begin{pmatrix} \theta_- \\ \theta_+ \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{im\sigma} & -e^{-im\sigma} \\ e^{im\sigma} & e^{-im\sigma} \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}, \tag{3.15}$$

which corresponds to (3.9). This change of variables leads to an expression with no σ-dependent exponential factors

$$\begin{aligned}\mathcal{L}_{2F} = & i\theta^+\partial_0\theta_+ + i\bar{\eta}\partial_0\eta + e^{i\mathcal{T}\tau}\partial_1\theta^+\bar{\eta} - e^{-i\mathcal{T}\tau}\partial_1\theta_+\eta - \mathcal{T}\bar{\eta}\eta \\ & +i\theta^-\partial_0\theta_- + i\eta^3\partial_0\eta_3 + e^{-i\mathcal{T}\tau}\eta^3\partial_1\theta^- - e^{i\mathcal{T}\tau}\eta_3\partial_1\theta_- + \mathcal{T}\eta^3\eta_3 \\ & +im(e^{-i\mathcal{T}\tau}\eta^3\theta^+ + e^{i\mathcal{T}\tau}\eta_3\theta_+ + e^{i\mathcal{T}\tau}\theta^-\bar{\eta} + e^{-i\mathcal{T}\tau}\theta_-\eta),\end{aligned} \tag{3.16}$$

where $\theta^+ = \theta_+^\dagger, \theta^- = \theta_-^\dagger$ and the winding-number dependence appears, which is compared with the case A. Under the shifts of $\theta_- \to e^{-i\mathcal{T}\tau}\theta_-, \theta_+ \to e^{i\mathcal{T}\tau}\theta_+$ the expression (3.16) becomes

$$\begin{aligned}\mathcal{L}_{2F} = & i\theta^+\partial_0\theta_+ + i\bar{\eta}\partial_0\eta - (\bar{\eta}\partial_1\theta^+ - \eta\partial_1\theta_+) - \mathcal{T}(\bar{\eta}\eta + \theta^+\theta_+) \\ & +i\theta^-\partial_0\theta_- + i\eta^3\partial_0\eta_3 + \eta^3\partial_1\theta^- - \eta_3\partial_1\theta_- + \mathcal{T}(\eta^3\eta_3 + \theta^-\theta_-) \\ & +im(e^{-2i\mathcal{T}\tau}\eta^3\theta^+ + e^{2i\mathcal{T}\tau}\eta_3\theta_+ + e^{2i\mathcal{T}\tau}\theta^-\bar{\eta} + e^{-2i\mathcal{T}\tau}\theta_-\eta).\end{aligned} \tag{3.17}$$

The winding-number dependent terms have a large time-dependent phase in the large limit of the total angular momentum so that they oscillate and average to zero as in [26, 28]. Therefore in the large $\mathcal{T}$ limit the m-dependent terms can be ignored. The resulting Lagrangian shows a simple separated expression in the same way as (3.12) of the case A.

Now for the case A we further set η_- and θ_3 to zero in (3.12) and (3.11) in order to obtain a fermionic Lagrangian with only two complex fermions η_+, θ

$$\mathcal{L}_F = i\eta^+\partial_0\eta_+ + i\bar{\theta}\partial_0\theta + \eta^+\partial_1\bar{\theta} - \eta_+\partial_1\theta - \mathcal{T}(\bar{\theta}\theta + \eta^+\eta_+). \tag{3.18}$$

Introducing a two-component complex (Dirac) spinor ψ by combining the two complex fermions as

$$\psi \equiv \begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix} = \begin{pmatrix}\eta_+ \\ \bar{\theta}\end{pmatrix} \text{ or } \begin{pmatrix}\theta \\ \eta^+\end{pmatrix} \tag{3.19}$$

we rewrite the fermionic Lagrangian (3.18) as

$$\mathcal{L}_F = i(\psi_1^\dagger\partial_0\psi_1 + \psi_2^\dagger\partial_0\psi_2) + \psi_1^\dagger\partial_1\psi_2 - \psi_2^\dagger\partial_1\psi_1 - \mathcal{T}(\psi_1^\dagger\psi_1 - \psi_2^\dagger\psi_2). \tag{3.20}$$

Further it takes a Lorentz-invariant expression for a Dirac fermion with mass $\mathcal{T}$ on the flat two-dimensional world-sheet

$$\mathcal{L}_F = i\bar{\psi}\rho^\mu\partial_\mu\psi + \mathcal{T}\bar{\psi}\psi \tag{3.21}$$

with $\rho^0 = -\sigma^3, \rho^1 = i\sigma^1$ and $\bar{\psi} = \psi^\dagger\rho^0$. This relativistic Lagrangian for the SU(1|2) sector is compared with the non-relativistic quadratic fluctuation Lagrangian of one complex fermion for the SU(1|1) sector produced by integrating an extra fermion in ref. [36].

In the fermionic action $S_F = \frac{\sqrt{\lambda}}{2\pi}\int d\tau d\sigma \mathcal{L}_F$ the overall factor $\sqrt{\lambda}$ can be removed by the rescalings of

$$\psi_\alpha \to \frac{\psi_\alpha}{\lambda^{1/4}} \quad (\alpha = 1, 2). \tag{3.22}$$

Moreover, the global AdS_5 time specified by $t = \mathcal{T}\tau$ in the large $\mathcal{T}$ limit yields

$$S_F = \int dt \int \frac{d\sigma}{2\pi}\left[i(\psi_1^\dagger\partial_0\psi_1 + \psi_2^\dagger\partial_0\psi_2) + \sqrt{\tilde{\lambda}}(\psi_1^\dagger\partial_1\psi_2 - \psi_2^\dagger\partial_1\psi_1) - (\psi_1^\dagger\psi_1 - \psi_2^\dagger\psi_2)\right], \tag{3.23}$$

where $\tilde{\lambda} = 1/\mathcal{T}^2 = \lambda/J^2$ is the effective BMN coupling constant. To create string states dual to the gauge theory operators in the SU(1|2) sector we need to choose a proper representation of the anti-commutation relations for fermions. The fermions are expanded in the Fourier modes

$$\psi_\alpha = \sum_{n=-\infty}^{\infty} e^{in\sigma}\psi_{\alpha,n}, \qquad \psi_\alpha^\dagger = \sum_{n=-\infty}^{\infty} e^{-in\sigma}\psi_{\alpha,n}^\dagger \tag{3.24}$$

by using the following creation and annihilation operators

$$\begin{pmatrix} \psi_{1n} \\ \psi_{2n} \end{pmatrix} = \begin{pmatrix} f_n & g_n \\ g_n & f_n \end{pmatrix}\begin{pmatrix} a_n^- \\ b_n^+ \end{pmatrix}, \qquad \begin{pmatrix} \psi_{1n}^\dagger \\ \psi_{2n}^\dagger \end{pmatrix} = \begin{pmatrix} f_n & -g_n \\ -g_n & f_n \end{pmatrix}\begin{pmatrix} a_n^+ \\ b_n^- \end{pmatrix}, \tag{3.25}$$

where the functions f_n, g_n are defined by

$$f_n = \sqrt{\frac{1}{2} + \frac{1}{2\omega_n}}, \qquad g_n = \frac{i\sqrt{\tilde{\lambda}}n}{1+\omega_n}\sqrt{\frac{1}{2} + \frac{1}{2\omega_n}} \tag{3.26}$$

with $\omega_n = \sqrt{1+\tilde{\lambda}n^2}$. The rotation matrices in (3.25) also take SU(2) forms. The substitution of (3.24) into the action (3.23) yields

$$S_F = \int dt \sum_{n=-\infty}^{\infty} [i(a_n^+\partial_0 a_n^- + b_n^+\partial_0 b_n^-) - \omega_n(a_n^+ a_n^- + b_n^+ b_n^-)], \tag{3.27}$$

which shows that (a^-, a^+) and (b^-, b^+) are pairs of canonically conjugate fermionic operators and ω_n is the energy of a plane-wave state. The long SYM operators in the SU(1|2) sector are dual to states obtained by acting operators $a_n^\dagger$ on the vacuum and switching off the b oscillators. Thus in the case of the non-point-like circular string background specified by the non-zero winding numbers we have produced the BMN-type plane-wave Hamiltonian for the SU(1|2) sector, as a collection of free massive fermionic oscillators. The Lagrangian (3.27) shows the similar oscillator expression to the quadratic plane-wave Lagrangian for the SU(1|1) sector in ref. [40], where the superstring Hamiltonian with the near-plane wave correction is constructed by using the uniform gauge and parametrizing the supercoset element in the different way from the one used in [38].

For the case B switching off θ_- and η_3 for (3.17) and (3.14) we have a reduced Lagrangian

$$\mathcal{L}_F = i\theta^+\partial_0\theta_+ + i\bar{\eta}\partial_0\eta - (\bar{\eta}\partial_1\theta^+ - \eta\partial_1\theta_+) - \mathcal{T}(\bar{\eta}\eta + \theta^+\theta_+). \tag{3.28}$$

In terms of the two complex fermions ψ_1, ψ_2 defined by

$$\psi \equiv \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \theta_+ \\ \bar{\eta} \end{pmatrix} \text{ or } \begin{pmatrix} \eta \\ \theta^+ \end{pmatrix} \tag{3.29}$$

the Lagrangian is expressed as

$$\mathcal{L}_F = i(\psi_1^\dagger\partial_0\psi_1 + \psi_2^\dagger\partial_0\psi_2) - \psi_1^\dagger\partial_1\psi_2 + \psi_2^\dagger\partial_1\psi_1 - \mathcal{T}(\psi_1^\dagger\psi_1 - \psi_2^\dagger\psi_2). \tag{3.30}$$

If we rename the world-sheet coordinate σ as $\sigma \to -\sigma$ the Lagrangian again takes the same relativistic expression for the Dirac fermion ψ as (3.21). The fermions $\psi_\alpha, \psi_\alpha^\dagger$ are expanded in the Fourier modes in the same way as (3.24) and (3.25), where f_n and g_n are now given by

$$f_n = \sqrt{\frac{1}{2} + \frac{1}{2\omega_n}}, \qquad g_n = \frac{-i\sqrt{\tilde{\lambda}}n}{1+\omega_n}\sqrt{\frac{1}{2} + \frac{1}{2\omega_n}}, \qquad \omega_n = \sqrt{1+\tilde{\lambda}n^2}, \tag{3.31}$$

which are obtained from (3.26) by $n \to -n$. The substitution of this mode expansion into the action for (3.30) leads to the same plane-wave action as (3.27).

4. Fermionic fluctuations over a circular string with two unqual spins

Let us consider a circular string background with two unequal spins J_1, J_2 and winding numbers m_1, m_2 [7]

$$X_i = a_i e^{i\omega_i \tau + i m_i \sigma}, \qquad \omega_i^2 = m_i^2 + \nu^2, \qquad \sum_{i=1}^{2} a_i^2 = 1, \tag{4.1}$$

whose classical energy $\mathcal{E}$ and $\mathcal{T}_i = \omega_i a_i^2$ $(i = 1, 2)$ are characterized by

$$\mathcal{E}^2 = 2\sum_{i=1}^{2} \omega_i \mathcal{T}_i - \nu^2, \qquad \sum_{i=1}^{2} m_i \mathcal{T}_i = 0. \tag{4.2}$$

For the case A we substitute the bosonic background solution (4.1) into the fermionic Lagrangian (2.4) and (2.5) with $\kappa/\sqrt{2} = 1$ to have

$$\begin{aligned}
\mathcal{L}_{2F} &= i\sum_{i=1}^{2} \eta^i \partial_0 \eta_i + i\theta^3 \partial_0 \theta_3 + i\bar{\theta}\partial_0\theta \\
&\quad + (a_1 e^{-i\delta_1}\eta^2 - a_2 e^{-i\delta_2}\eta^1)\partial_1\theta^3 - (a_1 e^{i\delta_1}\eta_2 - a_2 e^{i\delta_2}\eta_1)\partial_1\theta_3 \\
&\quad + (a_1 e^{i\delta_1}\eta^1 + a_2 e^{i\delta_2}\eta^2)\partial_1\bar{\theta} - (a_1 e^{-i\delta_1}\eta_1 + a_2 e^{-i\delta_2}\eta_2)\partial_1\theta \\
&\quad + (\omega_1 a_1^2 - \omega_2 a_2^2)(\eta_1\eta^1 - \eta_2\eta^2) \\
&\quad + a_1 a_2(\omega_1 + \omega_2)(e^{-i\delta_1 + i\delta_2}\eta_1\eta^2 + e^{i\delta_1 - i\delta_2}\eta_2\eta^1), \\
\mathcal{L}_{4F} &= -4\eta_1\eta^1\eta_2\eta^2,
\end{aligned} \tag{4.3}$$

where $\delta_i = \omega_i \tau + m_i \sigma$ $(i = 1, 2)$.

Performing the following SU(2)-type rotation of fermionic variables suggested from the terms including the σ-derivative in (4.3)

$$\begin{pmatrix} \eta_- \\ \eta_+ \end{pmatrix} = \begin{pmatrix} a_2 e^{i\delta_2} & -a_1 e^{i\delta_1} \\ a_1 e^{-i\delta_1} & a_2 e^{-i\delta_2} \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}, \tag{4.4}$$

we rewrite the Lagrangian (4.3) as

$$\begin{aligned}
\mathcal{L}_{2F} &= i\eta^+\partial_0\eta_+ + i\eta^-\partial_0\eta_- + i\theta^3\partial_0\theta_3 + i\bar{\theta}\partial_0\theta \\
&\quad - \eta^-\partial_1\theta^3 + \eta_-\partial_1\theta_3 - \eta^+\partial_1\bar{\theta} - \eta_+\partial_1\theta + 2(\omega_1 a_1^2 + \omega_2 a_2^2)(\eta_+\eta^+ - \eta_-\eta^-) \\
&\quad + 2a_1 a_2(\omega_1 - \omega_2)(e^{i(\delta_1+\delta_2)}\eta_+\eta^- + e^{-i(\delta_1+\delta_2)}\eta_-\eta^+), \\
\mathcal{L}_{4F} &= -4\eta^+\eta_+\eta^-\eta_-.
\end{aligned} \tag{4.5}$$

Under the redefinition the coupling terms have a phase and the mass terms are arranged to have a suitable mass parameter $\omega_1 a_1^2 + \omega_2 a_2^2 = \mathcal{T}_1 + \mathcal{T}_2 = \mathcal{T}$. The large angular momentum expansion for (4.1) gives $\nu^2 = \mathcal{T}^2 - \sum_{i=1}^2 m_i^2 \mathcal{T}_i/\mathcal{T} + \cdots$ and

$$\omega_1 = \mathcal{T} + \frac{(m_1^2 - m_2^2)\mathcal{T}_2}{2\mathcal{T}^2} + \cdots, \qquad \omega_2 = \mathcal{T} - \frac{(m_1^2 - m_2^2)\mathcal{T}_1}{2\mathcal{T}^2} + \cdots. \tag{4.6}$$

For the large total angular momentum $\mathcal{T}$ that means large ω_i ($i = 1, 2$), the coupling terms with the time-dependent phase can be ignored since they average to zero. Then we have a simple diagonalized expression

$$\begin{aligned}\mathcal{L}_{2F} = & \, i\eta^+\partial_0\eta_+ + i\bar{\theta}\partial_0\theta + \eta^+\partial_1\bar{\theta} - \eta_+\partial_1\theta - 2\mathcal{T}\eta^+\eta_+ \\ & + i\eta^-\partial_0\eta_- + i\theta^3\partial_0\theta_3 - \eta^-\partial_1\theta^3 + \eta_-\partial_1\theta_3 + 2\mathcal{T}\eta^-\eta_-,\end{aligned} \tag{4.7}$$

which is compared with (3.10). Here making the rescalings of fermions $\eta_+ \to e^{-i\mathcal{T}\tau}\eta_+, \eta_- \to e^{i\mathcal{T}\tau}\eta_-$, we observe that $\mathcal{L}_{4F}$ is not changed and $\mathcal{L}_{2F}$ is transformed into the same expression as (3.10). The rescalings combine with the τ- and σ-dependent rotation (4.4) into an SU(2)-type rotation

$$\begin{pmatrix} \eta_- \\ \eta_+ \end{pmatrix} = \begin{pmatrix} a_2 e^{-i\mathcal{T}\tau + i\delta_2} & -a_1 e^{-i\mathcal{T}\tau + i\delta_1} \\ a_1 e^{i\mathcal{T}\tau - i\delta_1} & a_2 e^{i\mathcal{T}\tau - i\delta_2} \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}, \tag{4.8}$$

which reduces to the previous σ-dependent rotation (3.9) for the equal spin case $\omega_1 = \omega_2 = \mathcal{T}, m_1 = -m_2 = -m, a_1 = a_2 = 1/\sqrt{2}$. Therefore the compensating redefinitions of $\theta \to e^{i\mathcal{T}\tau}\theta, \theta_3 \to e^{-i\mathcal{T}\tau}\theta_3$ for the transformed $\mathcal{L}_{2F}$ yield the same symmetric separated expression as (3.12). Thus for the case A reduced with $\eta_- = \theta_3 = 0$ in the circular string background with two unequal spins the quantum plane-wave spectrum (3.27) can be reproduced again through the suitable renaming of the fermions. The fermionic string configuration with the a oscillator only is considered to correpond to the long SYM operator which consists of the large and different number of two complex scalar fields and a few fermions in the SU(1|2) sector.

Now we turn our attention to the case B. The substitution of circular string solution (4.1) into the fermionic Lagrangian (2.4) and (2.5) with $\kappa/\sqrt{2} = 1$ gives again (3.14) for $\mathcal{L}_{4F}$ and an involved expression for $\mathcal{L}_{2F}$ which suggests the following particular choice of field redefinitions

$$\begin{pmatrix} \theta_- \\ \theta_+ \end{pmatrix} = \begin{pmatrix} a_2 e^{i\delta_2} & -a_1 e^{i\delta_1} \\ a_1 e^{-i\delta_1} & a_2 e^{-i\delta_2} \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}, \tag{4.9}$$

which has the same transformation matrix as (4.4) amd resembles (3.15). The inversion of (4.9) leads to the quadratic Lagrangian

$$\begin{aligned}\mathcal{L}_{2F} = & \, i\theta^+\partial_0\theta_+ + i\bar{\eta}\partial_0\eta - (\bar{\eta}\partial_1\theta^+ - \eta\partial_1\theta_+) - \mathcal{T}(\bar{\eta}\eta + \theta^+\theta_+) + im_a(\bar{\eta}\theta^+ + \eta\theta_+) \\ & + i\theta^-\partial_0\theta_- + i\eta^3\partial_0\eta_3 + \eta^3\partial_1\theta^- - \eta_3\partial_1\theta_- + \mathcal{T}(\eta^3\eta_3 + \theta^-\theta_-) + im_a(\eta^3\theta^- + \eta_3\theta_-) \\ & - a_1 a_2(\omega_1 - \omega_2)(e^{-i(\delta_1+\delta_2)}\theta^+\theta_- + e^{i(\delta_1+\delta_2)}\theta^-\theta_+) \\ & + ia_1 a_2(m_1 - m_2)[e^{i(\delta_1+\delta_2)}(\bar{\eta}\theta^- - \eta_3\theta_+) + e^{-i(\delta_1+\delta_2)}(\eta\theta_- - \eta^3\theta^+)] \\ & - a_1 a_2(\omega_1 - \omega_2)(e^{-i(\delta_1+\delta_2)}\eta\eta^3 + e^{i(\delta_1+\delta_2)}\eta_3\bar{\eta})\end{aligned} \tag{4.10}$$

with $m_a = m_1 a_1^2 + m_2 a_2^2$ which is compared with $\mathcal{T} = \omega_1 a_1^2 + \omega_2 a_2^2$. It is confirmed that this expression indeed reduces to (3.17) for the equal spin case. In this case the SU(2)-type rotation (4.9) can be expressed as

$$\begin{pmatrix} \theta_- e^{-i\mathcal{T}\tau} \\ \theta_+ e^{-i\mathcal{T}\tau} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{im\sigma} & -e^{-im\sigma} \\ e^{im\sigma} & e^{-im\sigma} \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}, \tag{4.11}$$

which is just the product of the previous σ-dependent rotation (3.15) and the succeeding shifts of $\theta_- \to e^{-i\mathcal{T}\tau}\theta_-, \theta_+ \to e^{i\mathcal{T}\tau}\theta_+$. The quartic Lagrangian $\mathcal{L}_{4F}$ remains the same expression as (3.14). Taking the fast-string limit of the quadratic Lagrangian $\mathcal{L}_{2F}$ (4.10) we see that the involved coupling terms with the exponential phase factors $e^{\pm i(\delta_1+\delta_2)}$ average to zero. In the resulting $\mathcal{L}_{2F}$ the fermionic system (θ_+, η) is separated from the fermionic one (θ_-, η_3), and each system has the non-zero winding-number dependent terms with a coefficient m_a, which vanish for the equal spin case.

Here putting $\theta_- = \eta_3 = 0$ for the reduction which keeps only two complex fermions θ_+, η, we have

$$\mathcal{L}_F = i\theta^+\partial_0\theta_+ + i\bar{\eta}\partial_0\eta - (\bar{\eta}\partial_1\theta^+ - \eta\partial_1\theta_+) - \mathcal{T}(\bar{\eta}\eta + \theta^+\theta_+) + im_a(\bar{\eta}\theta^+ + \eta\theta_+). \tag{4.12}$$

Through the renaming of fermionic fields in the same way as (3.29) $\mathcal{L}_F$ becomes

$$\mathcal{L}_F = i(\psi_1^\dagger\partial_0\psi_1 + \psi_2^\dagger\partial_0\psi_2) - \psi_1^\dagger(\partial_1 \pm im_a)\psi_2 + \psi_2^\dagger(\partial_1 \pm im_a)\psi_1 - \mathcal{T}(\psi_1^\dagger\psi_1 - \psi_2^\dagger\psi_2), \tag{4.13}$$

which is expressed through the renaming of $\sigma \to -\sigma$ as

$$\mathcal{L}_F = i\bar{\psi}\rho^\mu\partial_\mu\psi + \mathcal{T}\bar{\psi}\psi \pm m_a\bar{\psi}\rho^1\psi, \tag{4.14}$$

where $+$ corresponds to the one choice $\psi = \binom{\theta_+}{\bar{\eta}}$ and $-$ to the other choice $\psi = \binom{\eta}{\theta^+}$, and the bosonic background dependence is specified by $\mathcal{T}$ and m_a. We substitute the mode expansion of $\psi_\alpha, \psi_\alpha^\dagger$, (3.24) with (3.25) into the fermionic action for (4.13). If the following parametrization is chosen

$$f_n = \sqrt{\frac{1}{2} + \frac{1}{2\omega_n}}, \qquad g_n = \frac{-i\sqrt{\tilde{\lambda}}(n \pm m_a)}{1+\omega_n}\sqrt{\frac{1}{2} + \frac{1}{2\omega_n}}, \qquad \omega_n = \sqrt{1 + \tilde{\lambda}(n \pm m_a)^2}, \tag{4.15}$$

the same plane-wave action as (3.27) is derived. For the two unequal spin case we have observed that there seems a difference for the energy spectrum ω_n of each mode between the reduced A system (η_+, θ) and the reduced B system (θ_+, η). The ω_n for the reduced B system is specified with the mode number n shifted by the winding-number dependent factor m_a. However, using (4.6) we estimate m_a in the $\tilde{\lambda} = 1/\mathcal{T}^2$ expansion as

$$m_a = \frac{m_1\mathcal{T}_1}{\omega_1} + \frac{m_2\mathcal{T}_2}{\omega_2} = \frac{m_1\mathcal{T}_1 + m_2\mathcal{T}_2}{\mathcal{T}} - \frac{1}{\mathcal{T}^2}\frac{(m_1+m_2)(m_1-m_2)^2\mathcal{T}_1\mathcal{T}_2}{2\mathcal{T}^2} + \cdots, \tag{4.16}$$

whose first leading term is zero through (4.2). To the leading order in $\tilde{\lambda}$, that is, in the plane-wave limit both the reduced A system and the reduced B system show the same energy spectrum as Fock-space states for the fermionic fluctuation around the circular string background with two unequal spins.

For the "SU(2|2)" string theory sector we make a system by combining two cases A and B, that is, putting $X_3 = 0$ only in the fermionic Lagrangian $\mathcal{L}_{2F}$ (2.4) and $\mathcal{L}_{4F}$ (2.5), and take the large total angular momentum limit for the sum of the quadratic fermionic Lagrangians (4.7) and (4.10). The quartic Lagrangian becomes $\mathcal{L}_{4F} = \eta^+\eta_+\bar{\eta}\eta$ when the reduction specified by both $\eta_- = \theta_3 = 0$ and $\eta_3 = \theta_- = 0$ is taken. However, owing to the rescalings of fermionic fields by $1/\lambda^{1/4}$ such as (3.22) the quartic action becomes of the order $1/\sqrt{\lambda}$. In the leading large $\sqrt{\lambda}$ approximation near the classical bosonic solution with two spins we need to know only the quadratic part. The reduced quadratic action is shown to give the plane-wave spectra for the two fermions in the SU(2|2) sector by switching off the two relevant b oscillators.

5. Conclusion

From the $\mathrm{SU}(3) \times \mathrm{U}(1)$ invariant superstring action in $AdS_5 \times S^5$ which is produced by choosing the conformal gauge for the κ-symmetry gauge fixed superstring action, we have constructed the truncated action for the SU(1|2) sector which describes the fermionic fluctuations over a circular string background with two angular momenta and two winding numbers.

The suitable SU(2)-type τ- and σ-dependent rotation of fermionic fields and the first-string limit simplify the starting superstring action such that the involved coupling terms can be ignored and there remain two separated massive fermion systems. The mass terms have been characterized by the total angular momentum that arises from an adequate combination of the coupling terms between the fermionic fluctuations and the bosonic background fields. We have observed that the appropriate renamings of fermionic fields lead to a Lorentz-invariant action for a massive 2d Dirac fermion and its plane-wave spectrum for the SU(1|2) sector. The BMN-type spectrum has been derived even for the non-point-like string background with the non-zero winding numbers. For the truncated systems A and B for the two unequal spin case there appears a difference in the plane-wave spectra, but the difference specified by the winding numbers can be neglected to the leading order in $\tilde{\lambda}$. Combining the two truncated systems we have also constructed the leading plane-wave action for the SU(2|2) sector.

In the second and third references of [5] starting from the quadratic part of the covariant κ-symmetric superstring action [37], all the fermionic fluctuations over the multi-spin circular string solutions with three spins ($J_1 = J_2 = J', J_3 = J$) and two equal spins ($J', J = 0$) have been computed, while we have started from the full expression of the light-cone κ-symmetry gauge fixed superstring action [38] and extracted the spectrum of the truncated fermionic fluctuations in the SU(1|2) sector.

Recently in [44] in the uniform light-cone gauge the bosonic and fermionic quantum fluctuation spectra have been constructed in the near plane-wave limit for the SU(1|2) and SU(2|3) sectors, whose string configuration with one large angular momentum and no winding numbers is dual to the long SYM operator with a few bosonic impurity W fields and a few fermionic fields in a large number of bosonic Z fields, while our string configuration with two large angular momenta and two winding numbers for the SU(1|2) and SU(2|2) sectors is dual to the long SYM operator with a large number of bosonic impurity W fields and a few fermionic fields in a large number of bosonic Z fields. In spite of such differences for the bosonic backgrounds and the gauge choices, both fermionic leading spectra show the same BMN-type behavior in the limit of the large angular momenta.

References

[1] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200];
S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [hep-th/9802109];
E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150].

[2] R.R. Metsaev, *Type IIB Green-Schwarz superstring in plane wave Ramond-Ramond background*, *Nucl. Phys.* **B 625** (2002) 70 [hep-th/0112044];
R.R. Metsaev and A.A. Tseytlin, *Exactly solvable model of superstring in plane wave Ramond-Ramond background*, *Phys. Rev.* **D 65** (2002) 126004 [hep-th/0202109].

[3] D. Berenstein, J.M. Maldacena and H. Nastase, *Strings in flat space and pp waves from* $N = 4$ *super Yang-Mills*, *JHEP* **04** (2002) 013 [hep-th/0202021].

[4] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *A semi-classical limit of the gauge/string correspondence*, *Nucl. Phys.* **B 636** (2002) 99 [hep-th/0204051].

[5] S. Frolov and A.A. Tseytlin, *Semiclassical quantization of rotating superstring in* $AdS_5 \times S^5$, *JHEP* **06** (2002) 007 [hep-th/0204226]; *Multi-spin string solutions in* $AdS_5 \times S^5$, *Nucl. Phys.* **B 668** (2003) 77 [hep-th/0304255]; *Quantizing three-spin string solution in* $AdS_5 \times S^5$, *JHEP* **07** (2003) 016 [hep-th/0306130]; *Rotating string solutions: AdS/CFT duality in non-supersymmetric sectors*, *Phys. Lett.* **B 570** (2003) 96 [hep-th/0306143].

[6] N. Beisert, J.A. Minahan, M. Staudacher and K. Zarembo, *Stringing spins and spinning strings*, *JHEP* **09** (2003) 010 [hep-th/0306139];
N. Beisert, S. Frolov, M. Staudacher and A.A. Tseytlin, *Precision spectroscopy of AdS/CFT*, *JHEP* **10** (2003) 037 [hep-th/0308117];
G. Arutyunov, S. Frolov, J. Russo and A.A. Tseytlin, *Spinning strings in* $AdS_5 \times S^5$ *and integrable systems*, *Nucl. Phys.* **B 671** (2003) 3 [hep-th/0307191].

[7] G. Arutyunov, J. Russo and A.A. Tseytlin, *Spinning strings in* $AdS_5 \times S^5$*: new integrable system relations*, *Phys. Rev.* **D 69** (2004) 086009 [hep-th/0311004].

[8] A.A. Tseytlin, *Spinning strings and AdS/CFT duality*, hep-th/0311139.

[9] J.A. Minahan and K. Zarembo, *The Bethe-ansatz for* $N = 4$ *super Yang-Mills*, *JHEP* **03** (2003) 013 [hep-th/0212208].

[10] N. Beisert, *The complete one-loop dilatation operator of* $N = 4$ *super Yang-Mills theory*, *Nucl. Phys.* **B 676** (2004) 3 [hep-th/0307015];
N. Beisert and M. Staudacher, *The* $N = 4$ *SYM integrable super spin chain*, *Nucl. Phys.* **B 670** (2003) 439 [hep-th/0307042].

[11] N. Beisert, *The* SU(2|3) *dynamic spin chain*, *Nucl. Phys.* **B 682** (2004) 487 [hep-th/0310252].

[12] N. Beisert, *The dilatation operator of $N = 4$ super Yang-Mills theory and integrability*, *Phys. Rept.* **405** (2005) 1 [hep-th/0407277];
K. Zarembo, *Semiclassical Bethe ansatz and AdS/CFT*, *Comptes Rendus Physique* **5** (2004) 1081 [hep-th/0411191];
J. Plefka, *Spinning strings and integrable spin chains in the AdS/CFT correspondence*, hep-th/0507136.

[13] V.A. Kazakov, A. Marshakov, J.A. Minahan and K. Zarembo, *Classical/quantum integrability in AdS/CFT*, *JHEP* **05** (2004) 024 [hep-th/0402207].

[14] V.A. Kazakov and K. Zarembo, *Classical/quantum integrability in non-compact sector of AdS/CFT*, *JHEP* **10** (2004) 060 [hep-th/0410105];
N. Beisert, V.A. Kazakov and K. Sakai, *Algebraic curve for the* SO(6) *sector of AdS/CFT*, *Commun. Math. Phys.* **263** (2006) 611 [hep-th/0410253];
S. Schäfer-Nameki, *The algebraic curve of 1-loop planar $N = 4$ SYM*, *Nucl. Phys.* **B 714** (2005) 3 [hep-th/0412254];
N. Beisert, V.A. Kazakov, K. Sakai and K. Zarembo, *The algebraic curve of classical superstrings on $AdS_5 \times S^5$*, *Commun. Math. Phys.* **263** (2006) 659 [hep-th/0502226].

[15] N. Beisert, V. Dippel and M. Staudacher, *A novel long range spin chain and planar $N = 4$ super Yang-Mills*, *JHEP* **07** (2004) 075 [hep-th/0405001].

[16] G. Arutyunov, S. Frolov and M. Staudacher, *Bethe ansatz for quantum strings*, *JHEP* **10** (2004) 016 [hep-th/0406256].

[17] C.G. Callan Jr. et al., *Quantizing string theory in $AdS_5 \times S^5$: beyond the pp-wave*, *Nucl. Phys.* **B 673** (2003) 3 [hep-th/0307032];
C.G. Callan Jr., T. McLoughlin and I.J. Swanson, *Holography beyond the Penrose limit*, *Nucl. Phys.* **B 694** (2004) 115 [hep-th/0404007]; *Higher impurity AdS/CFT correspondence in the near-BMN limit*, *Nucl. Phys.* **B 700** (2004) 271 [hep-th/0405153].

[18] A. Rej, D. Serban and M. Staudacher, *Planar $N = 4$ gauge theory and the Hubbard model*, *JHEP* **03** (2006) 018 [hep-th/0512077].

[19] M. Staudacher, *The factorized S-matrix of CFT/AdS*, *JHEP* **05** (2005) 054 [hep-th/0412188].

[20] N. Beisert and M. Staudacher, *Long-range* PSU(2, 2|4) *Bethe ansaetze for gauge theory and strings*, *Nucl. Phys.* **B 727** (2005) 1 [hep-th/0504190].

[21] C.G. Callan Jr., J. Heckman, T. McLoughlin and I.J. Swanson, *Lattice super Yang-Mills: a virial approach to operator dimensions*, *Nucl. Phys.* **B 701** (2004) 180 [hep-th/0407096].

[22] N. Beisert, *The* SU(2|2) *dynamic S-matrix*, hep-th/0511082;
An SU(1|1)*-invariant S-matrix with dynamic representations*, hep-th/0511013;
R.A. Janik, *The $AdS_5 \times S^5$ superstring worldsheet S-matrix and crossing symmetry*, *Phys. Rev.* **D 73** (2006) 086006 [hep-th/0603038].

[23] B.I. Zwiebel, *$N = 4$ SYM to two loops: compact expressions for the non-compact symmetry algebra of the* SU(1, 1|2) *sector*, *JHEP* **02** (2006) 055 [hep-th/0511109].

[24] S.A. Frolov, I.Y. Park and A.A. Tseytlin, *On one-loop correction to energy of spinning strings in S^5*, *Phys. Rev.* **D 71** (2005) 026006 [hep-th/0408187];
I.Y. Park, A. Tirziu and A.A. Tseytlin, *Spinning strings in $AdS_5 \times S^5$: one-loop correction to energy in* SL(2) *sector*, *JHEP* **03** (2005) 013 [hep-th/0501203].

JHEP08(2006)028

[25] S. Schäfer-Nameki, M. Zamaklar and K. Zarembo, *Quantum corrections to spinning strings in $AdS_5 \times S^5$ and Bethe ansatz: a comparative study*, *JHEP* **09** (2005) 051 [hep-th/0507189];
S. Schäfer-Nameki and M. Zamaklar, *Stringy sums and corrections to the quantum string Bethe ansatz*, *JHEP* **10** (2005) 044 [hep-th/0509096];
N. Beisert and A.A. Tseytlin, *On quantum corrections to spinning strings and Bethe equations*, *Phys. Lett.* **B 629** (2005) 102 [hep-th/0509084];
S. Schäfer-Nameki, *Exact expressions for quantum corrections to spinning strings*, *Phys. Lett.* **B 639** (2006) 571 [hep-th/0602214].

[26] A. Mikhailov, *Speeding strings*, *JHEP* **12** (2003) 058 [hep-th/0311019]; *Slow evolution of nearly-degenerate extremal surfaces*, *J. Geom. Phys.* **54** (2005) 228 [hep-th/0402067]; *Supersymmetric null-surfaces*, *JHEP* **09** (2004) 068 [hep-th/0404173]; *Notes on fast moving strings*, *Commun. Math. Phys.* **264** (2006) 683 [hep-th/0409040].

[27] M. Kruczenski, *Spin chains and string theory*, *Phys. Rev. Lett.* **93** (2004) 161602 [hep-th/0311203].

[28] M. Kruczenski and A.A. Tseytlin, *Semiclassical relativistic strings in S^5 and long coherent operators in $N = 4$ SYM theory*, *JHEP* **09** (2004) 038 [hep-th/0406189].

[29] M. Kruczenski, A.V. Ryzhov and A.A. Tseytlin, *Large spin limit of $AdS_5 \times S^5$ string theory and low energy expansion of ferromagnetic spin chains*, *Nucl. Phys.* **B 692** (2004) 3 [hep-th/0403120];
R. Hernandez and E. Lopez, *The* SU(3) *spin chain sigma model and string theory*, *JHEP* **04** (2004) 052 [hep-th/0403139];
C. Kristjansen and T. Mansson, *The circular, elliptic three-spin string from the* SU(3) *spin chain*, *Phys. Lett.* **B 596** (2004) 265 [hep-th/0406176].

[30] B. Stefanski Jr. and A.A. Tseytlin, *Large spin limits of AdS/CFT and generalized Landau-Lifshitz equations*, *JHEP* **05** (2004) 042 [hep-th/0404133].

[31] S. Bellucci, P.Y. Casteill, J.F. Morales and C. Sochichiu, SL(2) *spin chain and spinning strings on $AdS_5 \times S^5$*, *Nucl. Phys.* **B 707** (2005) 303 [hep-th/0409086];
S. Ryang, *Circular and folded multi-spin strings in spin chain sigma models*, *JHEP* **10** (2004) 059 [hep-th/0409217].

[32] J.A. Minahan, A. Tirziu and A.A. Tseytlin, $1/J$ *corrections to semiclassical AdS/CFT states from quantum Landau-Lifshitz model*, *Nucl. Phys.* **B 735** (2006) 127 [hep-th/0509071]; $1/J^2$ *corrections to BMN energies from the quantum long range Landau-Lifshitz model*, *JHEP* **11** (2005) 031 [hep-th/0510080];
A. Tirziu, *Quantum Landau-lifshitz model at four loops: $1/J$ and $1/J^2$ corrections to BMN energies*, *Phys. Rev.* **D 73** (2006) 106001 [hep-th/0601139].

[33] R. Hernandez and E. Lopez, *Spin chain sigma models with fermions*, *JHEP* **11** (2004) 079 [hep-th/0410022].

[34] S. Bellucci, P.Y. Casteill and J.F. Morales, *Superstring sigma models from spin chains: the* $\mathrm{SU}(1,1|1)$ *case*, *Nucl. Phys.* **B 729** (2005) 163 [hep-th/0503159].

[35] S. Bellucci and P.Y. Casteill, *Sigma model from* $\mathrm{SU}(1,1|2)$ *spin chain*, *Nucl. Phys.* **B 741** (2006) 297 [hep-th/0602007].

[36] B. Stefanski Jr. and A.A. Tseytlin, *Super spin chain coherent state actions and $AdS_5 \times S^5$ superstring*, *Nucl. Phys.* **B 718** (2005) 83 [hep-th/0503185].

[37] R.R. Metsaev and A.A. Tseytlin, *Type IIB superstring action in $AdS_5 \times S^5$ background*, *Nucl. Phys.* **B 533** (1998) 109 [hep-th/9805028].

[38] R.R. Metsaev and A.A. Tseytlin, *Superstring action in $AdS_5 \times S^5$: κ-symmetry light cone gauge*, *Phys. Rev.* **D 63** (2001) 046002 [hep-th/0007036];
R.R. Metsaev, C.B. Thorn and A.A. Tseytlin, *Light-cone superstring in AdS space-time*, *Nucl. Phys.* **B 596** (2001) 151 [hep-th/0009171].

[39] R. Roiban and W. Siegel, *Superstrings on $AdS_5 \times S^5$ supertwistor space*, *JHEP* **11** (2000) 024 [hep-th/0010104];
I. Bena, J. Polchinski and R. Roiban, *Hidden symmetries of the $AdS_5 \times S^5$ superstring*, *Phys. Rev.* **D 69** (2004) 046002 [hep-th/0305116];
A. Das, J. Maharana, A. Melikyan and M. Sato, *The algebra of transition matrices for the $AdS_5 \times S^5$ superstring*, *JHEP* **12** (2004) 055 [hep-th/0411200];
L.F. Alday, G. Arutyunov and A.A. Tseytlin, *On integrability of classical superstrings in $AdS_5 \times S^5$*, *JHEP* **07** (2005) 002 [hep-th/0502240];
A. Das, A. Melikyan and M. Sato, *The algebra of flat currents for the string on $AdS_5 \times S^5$ in the light-cone gauge*, *JHEP* **11** (2005) 015 [hep-th/0508183].

[40] L.F. Alday, G. Arutyunov and S. Frolov, *New integrable system of 2dim fermions from strings on $AdS_5 \times S^5$*, *JHEP* **01** (2006) 078 [hep-th/0508140].

[41] G. Arutyunov and S. Frolov, *Uniform light-cone gauge for strings in $AdS_5 \times S^5$: solving* SU(1|1) *sector*, *JHEP* **01** (2006) 055 [hep-th/0510208].

[42] T. McLoughlin and I.J. Swanson, *N-impurity superstring spectra near the pp-wave limit*, *Nucl. Phys.* **B 702** (2004) 86 [hep-th/0407240].

[43] T. Klose and K. Zarembo, *Bethe ansatz in stringy sigma models*, *J. Stat. Mech.* (2006) P05006 [hep-th/0603039].

[44] S. Frolov, J. Plefka and M. Zamaklar, *The $AdS_5 \times S^5$ superstring in light-cone gauge and its Bethe equations*, hep-th/0603008.

JHEP08(2006)028